Bernoulli decompositions and applications

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Outline

Equidistributed sequences

Bernoulli systems

Sinai's factor theorem

A reminder

Let $\{x_n\}_{n\geq 1}$ be a sequence in $[0,1]$. It is *equidistributed* with respect to the Lebesgue measure λ if for each (open or close or whatever) interval $I \subset [0,1]$ we have the following result,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1_I(x_n) = \lambda(I).$$

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It is enough to check the above result for each interval with rational end points.
It is also enough to replace the indicator function with a countable dense family of continuous functions in $C([0,1])$.

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I will give you an equidistributed sequence $\{x_n\}_{n\geq 1}$ in $[0,1]$ and a small number ρ, say, $\rho = 0.001$. You must choose a subsequence K of $\mathbb{N}$ with upper density ρ and minimize the Lebesgue measure of $\overline{\{x_n\}_{n\in K}}$.

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Claim: This is a fair game. Or is it?

A result

Lemma

Let $\{x_n\}_{n\geq 1}$ be an equidistributed sequence in $[0,1]$. Let $K \subset \mathbb{N}$ be a sequence with upper density ρ. Then $\overline{\{x_n\}_{n\geq K}}$ has Lebesgue measure at least ρ.

A result

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Proof.

There is a proof but there is no space for writing it down. □

Choose a random subsequence

Lemma

Let $\{x_n\}_{n\geq 1}$ *be an equidistributed sequence in* $[0,1]$. *Let* $K \subset \mathbb{N}$ *be a sequence chosen randomly by including each integer* $k \in K$ *independently with probability* $p \in (0,1)$. *Then almost surely,* $\{x_n\}_{n\in K}$ *is dense in* $[0,1]$. *In fact, the new sequence equidistributes in* $[0,1]$ *if it is enumerated properly.*

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Lemma

Let $\{x_n\}_{n\geq 1}$ be an equidistributed sequence in $[0,1]$. Let $K \subset \mathbb{N}$ be a sequence chosen randomly by including each integer $k \in K$ independently with probability $p \in (0,1)$. Then almost surely, $\{x_n\}_{n\in K}$ is dense in $[0,1]$. In fact, the new sequence equidistributes in $[0,1]$ if it is enumerated properly.

Remark

This holds no matter how small p is.

Another reminder

Definition (Bernoulli system)

Let Λ be a finite set of digits. Consider the space $\Omega = \Lambda^{\mathbb{N}}$ equipped with the product topology and the cylinderical σ-algebra. Given a probability measure (vector) p on Λ we also define the product probability measure ν on Ω. Let $S : \Omega \to \Omega$ be the left shift. Then (Ω, S, ν) mixing and we call it a Bernoulli system.

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The measure theoretic entropy of (Ω, S, ν) is equal to

$$-\sum_{\lambda \in \Lambda} p_\lambda \log p_\lambda,$$

where $p_\lambda, \lambda \in \Lambda$ is the probability vector on Λ which gives the measure ν on Ω.

Theorem

Let $K \subset \mathbb{N}$ be a sequence with upper density ρ. Let $\Omega_1, \ldots, \Omega_M$ be pairwise disjoint measurable events. Suppose that $\Omega_1 \cup \cdots \cup \Omega_M$ has measure at least $1 - \epsilon$ and $\epsilon < \rho$. Then for ν almost all $\omega \in \Omega$, there is an index $i(\omega) \in \{1, \ldots, M\}$ such that $\overline{\{x_n\}_{n \in K \cap K_{\Omega_{i(\omega)}}(\omega)}}$ has Lebesgue measure at least $\rho - \epsilon$.

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Proof.

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Sinai's factor theorem

Definition (Factor)

A measurable dynamical system is in general denoted as $(X, \mathcal{X}, S, \mu)$ *where* X *is a set with* σ*-algebra* $\mathcal{X}$ *and measure* μ *and a measurable map* $S : X \to X$*. Given two dynamical systems* $(X, \mathcal{X}, S, \mu), (X_1, \mathcal{X}_1, S_1, \mu_1)$*, a measurable map* $f : X \to X_1$ *is called a factorization map and* $(X_1, \mathcal{X}_1, S_1, \mu_1)$ *is called a factor of* $(X, \mathcal{X}, S, \mu)$ *if* $\mu_1 = f\mu$ *and* $f \circ S = S_1 \circ f$*.*

Theorem

Given an ergodic dynamical system (X, T, μ) *with positive entropy* $h(T, \mu) > 0$*, any Bernoulli system* (Ω, S, ν) *with entropy* $h(S, \nu) \leq h(T, \mu)$ *is a factor of* (X, T, μ)*.*

A motivating exercise

Let α be an irrational number and d is an arbitrary real number. Try to compute the (upper/lower) box dimension of the sequence $\{n\alpha + 2^n d\}_{n\geq 1}$.

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Let α be an irrational number and d is an arbitrary real number. Try to compute the (upper/lower) box dimension of the sequence $\{n\alpha + 2^n d\}_{n\geq 1}$.
Hint: Consider the case for d being of 'zero entropy' and then use Sinai's factor theorem to treat the case for the 'positive entropy' case.

A challenging exercise

Let $L \subset [0, 1]$ be a compact set. Define the following *sparseness indicating sequence* of L around $a \in L$,

$$W(L, a) = \{k \in \mathbb{N} : \exists b \in L, |b - a| \in [2^{-k}, 2^{-k+1}]\}.$$

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Let $A_2, A_3 \subset [0, 1]$ be $\times 2, \times 3$ invariant closed sets respectively.
Show that $L = A_2 \cap A_3$ is sparse if $\dim_H A_2 + \dim_H A_3 < 1$.

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Remark

This implies that $\dim_H L = 0$. *(A recent result by Wu and by Shmerkin)*

Thanks.

P.S. The solutions of the exercises can be provided upon request.